Grade 9/10 Math Circles

February 8, 2023

The Shape of You - Problem Set Solutions

Pascal's Triangle

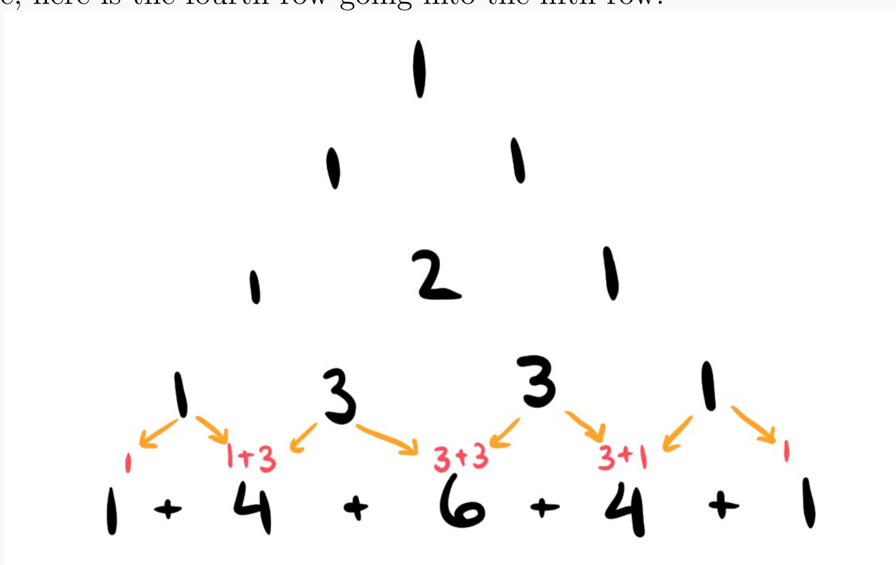
1. Prove that the sum of the entries in the n-th row of Pascal's Triangle equals 2^{n-1}. Isn't this a little weird?

Solution: The first row sums to $1 = 2^0$, so it suffices to show that the sum of the entries in each successive row is twice the sum of the entries in the previous row. This would show that the sums of the rows are $1, 2, 4, 8, 16, \ldots$ and so on, and we'd be done.

This fact actually follows quickly using the equation we proved in lecture

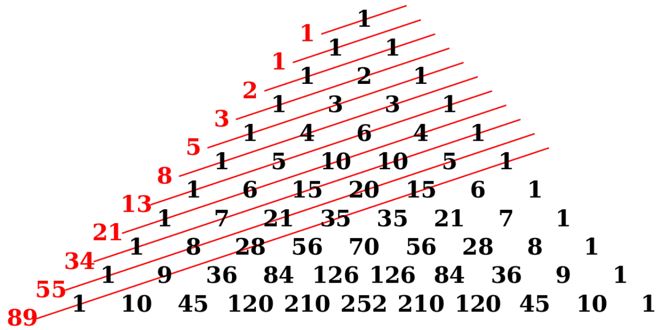
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

For example, here is the fourth row going into the fifth row:



Notice that there are exactly two copies of every number in the fourth row being added in the fifth row sum. The general case is argued the same way.

2. If you look at Pascal's Triangle for long enough, you might notice this pattern:



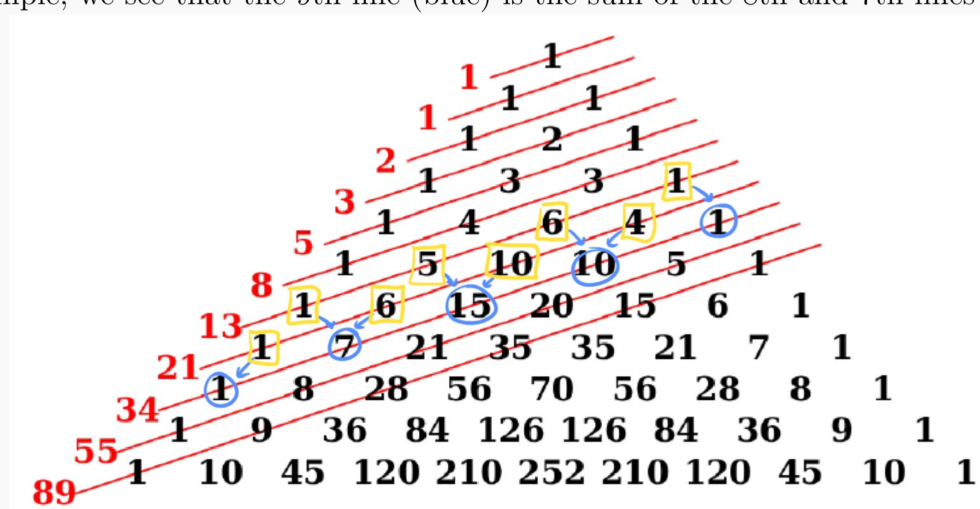
Give an argument for why the Fibonacci numbers magically appear here as the sums of the diagonals in Pascal's Triangle...

Solution: The Fibonacci sequence looks like $1, 1, 2, 3, 5, 8, 13, \ldots$ and so on. So it starts with $1, 1$, and after that every successive number in the sequence is the sum of the previous two. We see that the first two diagonals above do start off with $1, 1$. So it is enough to argue that the sum on each diagonal is equal to the sum of numbers on the previous two diagonals.

Again this follows quickly using the equation we proved in lecture:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

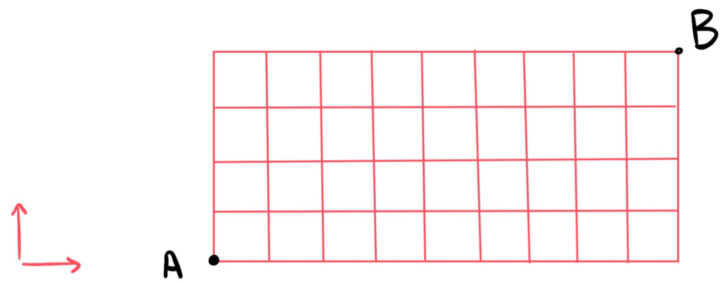
For example, we see that the 9th line (blue) is the sum of the 8th and 7th lines (yellow):

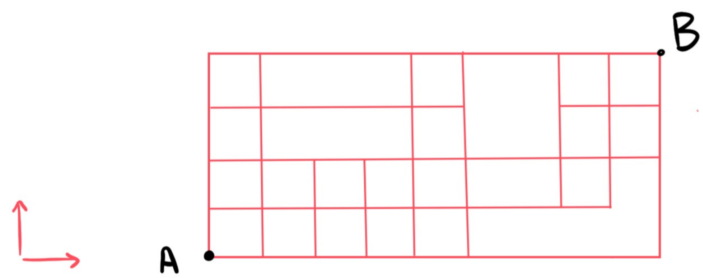


The general case should be clear from here, since the angle of the diagonal lines doesn't change (note: the numbers that each line intersect follow the same pattern as how a knight moves on a chess board).

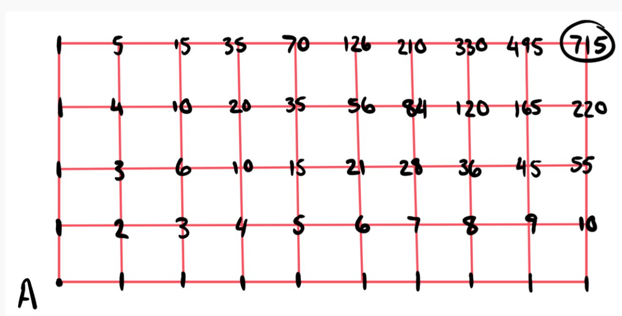
Path Counting

3. Count (and then defend your answer) the number of ways to get from point A to point B in the following diagrams. The only allowed directions of motions are indicated by the red arrows.

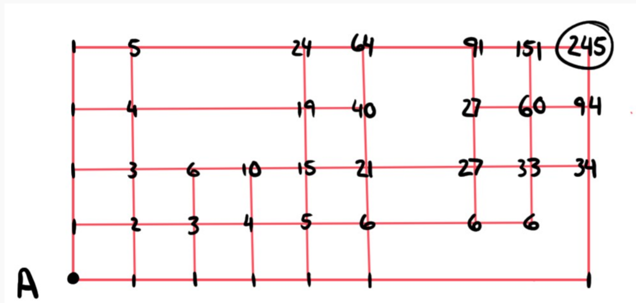




Solution: The first problem is just reconstructing (a rotated version of) Pascal's Triangle:



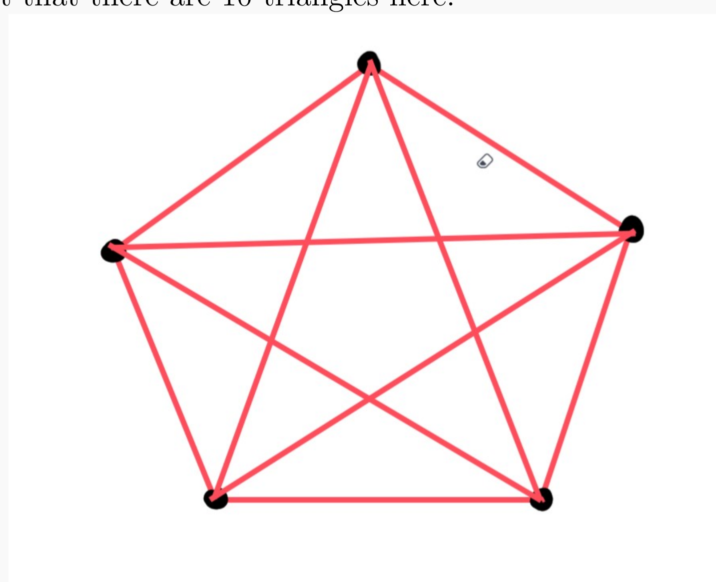
The second problem uses the same algorithm, but the weird spaces makes the numbers different. At each intersection, we count the number of paths by adding together the numbers in the intersection to the left and underneath (which both represent the number of paths to get to the intersection to the left and underneath, respectively). As a third year pure math student, I claim no correctness in my ability to add two-digit numbers together.



Combinations and Permutations

4. Suppose there are five friends who want to organize a committee of three people[1] Compute the number of way to do this, ie. the number of ways to choose 3 people (or objects) out of a set of 5. Note: it may be helpful to draw a 5-simplex.

Solution: Choosing 3 people out of a set of 5 to make a committee is the same as choosing 3 dots out of a 5-simplex to make a triangle. So this is the third number in the $(5+1)$-th row of Pascal's Triangle, which we already computed is 10. Here is a 5 simplex to confirm, and you can count that there are 10 triangles here:



[1]Why these people spend their time forming committees is beyond me.

5. Suppose now that you are choosing a 3-letter long password out of the 5 letters A,B,C,D,E, and you cannot use the same letter twice in the password (just like you could not have one person count as two people in the previous question). Compute the total number of possible passwords.

> *Solution*: Let's count! From the last question, we know that there are 10 ways to choose 3 things out of a set of 5, so there are 10 ways to choose 3 letters for the password. However, passwords are sensitive to the order in which letters are arranged. For example, ABC is certainly a different password from CBA. There are actually 6 different ways to arrange 3 letters in order: ABC, ACB, BAC, BCA, CAB, CBA. So there are $10 \times 6 = 60$ different passwords.

6. Compare your answers to the previous two questions. Did you get the same answer? Different answers? Why?

> *Solution*: The answers are different because, in the second question, the *order* of the elements being picked matters, unlike in the first question, where the order doesn't matter. So we have to count more, ie. the answer to the second question is bigger, because every time we pick 3 elements, we also have to consider all the different possible ways those elements can be arranged.

"Word" Problems

7. How many possible seven letter words are possible using the letters S, I, M, P, L, E, X? Define a word to be any sequence of letters. Hint: try calculating this for a smaller number of letters and look for a pattern. What if you had n letters instead of 7?

> *Solution*: There are 7 choices for the first letter, then 6 choices remaining for the second letter, then 5 choices remaining for the third letter, and so on. In total, that makes $7 \times 6 \times 5 \times \ldots \times 2 \times 1$ choices. For n letters, the arguments would be the same, and we use the notation $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$.

8. How many possible eight letter words are possible using the letters W, A, T, E, R, L, O, O? Note that two words are the same if they look like the same sequence of letters.

> *Solution*: Suppose first for simplicity that all the letters are distinct: so for example, we could label the letters W, A, T, E, R L, O_1, O_2. Then just as before, there are 7! possible words. But for each word, there is a word that "looks the same", because the O's are just swapped. For example, WATO_1ERO_2L should be the same as WATO_2ERO_1L. So we are counting twice as many words as we should be! So there are 7!/2 possible words.
>
> In general, whenever we have k copies of the same letter, then by labelling as above we see that there are $k!$ ways to arrange those copies if we treat each copy as distinct, so we can just count everything as if all the letters are distinct and then divide by $k!$.
>
> For another example, taking the letters W,A,R,R,I,O,R, then we can make 7!/3! possible different words because there are 3 repeated R's and therefore $3! = 6$ ways to rearrange those R's without actually changing a specific word.

9. Compare your answers to the previous two questions. Did you get the same answer? Different answers? Why?

> *Solution*: The second answer is twice as small because we had to account for the fact that there are two letters being used that are the same.

10. How many possible six letter words are possible using the letters N, N, C, C, C?

> *Solution*: From the work done above, we can first consider the letters as all being distinct N_1, N_2, C_1, C_2, C_3, giving 5! possible words, but we know this is overcounting. Then, accounting for the repeated letters (there are 2 repeated N's and 3 repeated C's), we actually get $5!/(2!3!) = 10$ possible words.

11. Compare your answer above to your answer in the (rather absurd) committee problem in question 4. Did you get the same answer? Different answers? Why?

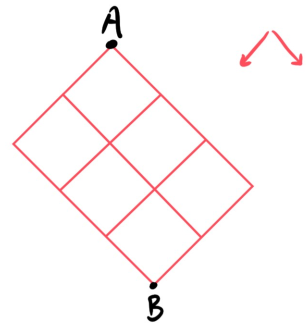
    Hint: I *coincidentally* always represent "chosen" by the letter C and represent "not chosen" by the letter N. Hm.

> *Solution*: It's the same answer! We can think of as making a committee as labelling each person with "Chosen" (C) and "Not chosen" (N). Then a possible committee would correspond to a specific sequence of N's and C's.
>
> For example, say that the friends are: Alice, Bob, Charlie, Dave, and Eve. If we choose Bob, Charlie, and Eve to make a committee, and then give each person a letter representing their status, then (keeping the same order of people), we'd get the word
>
> $$NCCNC$$

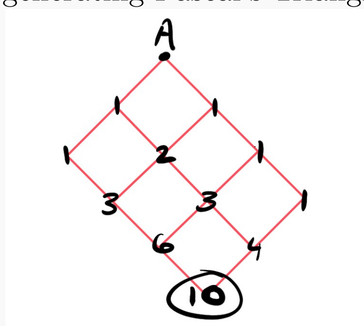
12. How many possible five letter words are possible using the letters L, L, R, R, R?

> *Solution*: Same as above: this should be $5!/(2!3!) = 10$. Note that the letters are different from before, but this doesn't change our answer, because what the actual letters are shouldn't matter (we only care about how many letters there are and how many copies of each there are).

13. Count the number of ways to get from point A to point B in the following diagrams. The only allowed directions of motions are indicated by arrows.



Solution: This is the same as generating Pascal's Triangle:



14. Compare your answers to the previous two questions. Did you get the same answer? Different answers? Why?

Hint: It *just so happens* that I always represent "left" by the letter L and represent "right" by the letter R. Hmmm.

Solution: The answers are the same! Notice that to get from point A to point B according to the allowed directions given by the red arrows, we will always have to make exactly 5 moves, and each move brings us closer to B. Now suppose each time we make a move, we write down if we went left (L) or right (R). For example, what is the path that corresponds to LRRLR?
Can you convince yourself that we will always have to make 5 moves, and 2 of those will have to be to the left, and 3 of those will have to be to the right? So it suffices to count the number of ways that we can choose 2 "moves" out of the 5 moves we make to be the left moves, and the other ones will be right moves by default. In other words, we are counting the number of possible words using L,L,R,R,R.

15. **Extra super hardcore brainpower challenge**: Use the above exercises to find and *prove* a closed form formula for $\binom{n}{k}$. *Closed form* means that the formula should look like some arithmetic operations between numbers and the variables n and k.

Culture: we already proved in lecture the *recursive* formula that $\binom{n}{0} = \binom{n}{n} = 1$ and

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

This is called a *recursive* formula because you need to recompute earlier smaller values and combine them before you get $\binom{n}{k}$. In particular, it isn't *closed form.*

Solution: From the above problems in this section, it should be clear to deduce that the number of ways to pick "k" objects out of a set of "n" possible objects is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

My favorite way to prove this is to list the n objects in a line, and label every chosen object with a C and label every not-chosen object with an N. So how many words can we make out of k copies of the letter C and $n-k$ copies of the letter N? Exactly as above.